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# ON JOINTLY ESTIMATING PARAMETERS AND MISSING DATA BY MAXIMIZING THE COMPLETE-DATA LIKELIHOOD

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## ABSTRACT

One approach to handling incomplete data occasionally encountered in the literature is to treat the missing data as parameters and to maximize the complete data likelihood over missing data and parameters. This paper points out that although this approach can be useful in particular problems, it is not a generally reliable approach to the analysis of incomplete data. In particular, it does not share the optimal properties of maximum likelihood estimation, except under the trivial asymptotics in which the proportion of missing data goes to zero as the sample size increases.



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# ON JOINTLY ESTIMATING PARAMETERS AND MISSING DATA BY MAXIMIZING THE COMPLETE-DATA LIKELIHOOD

Roderick J. A. Little and Donald B. Rubin

## 1. Introduction

In the standard formulation of maximum likelihood theory for complete data, the data $z$ are assumed to have a distribution with density $f(z|\theta)$ indexed by an unknown parameter $\theta$. Having observed data values $z = \tilde{z}$, the likelihood of $\theta$ is the density of the observed data regarded as a function of $\theta$, that is

$$L(\theta|\tilde{z}) = f(\tilde{z}|\theta) \quad \text{for all } \theta \ . \tag{1}$$

The maximum likelihood estimate $\hat{\theta}$ of $\theta$ is obtained by maximizing (1) with respect to $\theta$. We use the term complete data likelihood to refer to the expression (1).

Now suppose that some of the values in $z$ are not observed. Let $z_m$ denote the <u>m</u>issing components and $z_p$ the observed (<u>p</u>resent) components where $\tilde{z}_p$ is the observed value of $z_p$. It is not uncommon in the literature on incomplete data to see the suggestion that estimates of $\theta$ can be found by treating the missing values $z_m$ as parameters and maximizing the complete data likelihood with respect to $\theta$ and $z_m$. In symbols, this corresponds to maximizing the function

$$L_1(\theta, z_m|\tilde{z}_p) = f(z_m, \tilde{z}_p|\theta) \tag{2}$$

with respect to $(\theta, z_m)$. The classic example of this approach is in the analysis of missing plots in analysis of variance where missing outcomes $z_m$ are treated as parameters and then filled in to allow computationally efficient methods to be used for analysis (Anderson, 1946; Bartlett, 1937;

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Rubin, 1972). More recently, DeGroot and Goel (1980) propose this approach as one possibility for the analysis of a mixed up bivariate normal sample, where the missing data are the indices that allow the values of the two variables to be paired, and a priori all pairings are assumed equally likely. Press and Scott (1976) present a Bayesian analysis of an incomplete multivariate normal sample which is formally equivalent to maximizing (2). They maximize the joint posterior distribution of $\theta$ and $z_m$, after specifying a flat prior distribution the parameter $\theta$.

Although the literature on missing plot analysis explicitly recognizes the problems resulting from the suggested procedure, the more recent literature can be read as implying that maximizing (2) over missing data and parameters is just as principled as standard maximum likelihood estimation from the complete data. Our purpose is simply to point out the joint maximization over missing data and parameters is <u>not</u> a maximum likelihood procedure in any useful sense of the term. It does not in general enjoy the optimal large sample properties of maximum likelihood estimation, except using the trivial asymptotics in which the fraction of the data which are missing goes to zero as the sample size increases.

From the likelihood perspective, missing data $z_m$ differ fundamentally from parameters $\theta$ in that they are random variables with an a priori specified probability distribution. The correct likelihood is obtained by integrating the missing data $z_m$ out of the complete data likelihood (1), that is, the correct likelihood is

$$L_2(\theta|\tilde{z}_p) = \int f(z_m, \tilde{z}_p|\theta)dz_m, \quad \text{for all } \theta \quad . \tag{3}$$

This formulation implicitly assumes that the missing data are missing at random (Rubin, 1976). In particular, the probability that a value is missing does not depend on the missing data $z_m$, although it may depend on values

which are observed. If the missing data are not missing at random, then the model formulation needs to include a distribution for the set of variables indicating whether values are observed or missing. For details, see Rubin (1976).

Assuming the missing data are missing at random, $L_2$ given by (3) is equal to the probability density of the observed data $z_p$ regarded as a function of the unknown parameter, that is, of quantities not having a probability distribution. Hence $L_2$ and not $L_1$ is the true likelihood of $\theta$ given incomplete data $\tilde{z}_p$. In the next section we compare parameter estimates of $\theta$ found by maximizing $L_1$ with maximum likelihood estimates found by maximizing $L_2$ for some simple problems.

## 2. Examples

### Example 1. Univariate Normal Sample

Suppose that $z$ consists of $N$ observations from a Normal distribution with mean $\mu$ and variance $\sigma^2$, $z_p$ consists of $n$ observations which are observed and $z_m$ represents $N-n$ missing observations which are assumed missing at random. Let $\bar{z}$ and $s_z^2$ denote the sample mean and sample variance (with denominator $n$) of the $n$ observed values. Then $\theta = (\mu, \sigma^2)$, and maximizing $L_2$ leads to maximum likelihood estimates $\mu = \bar{z}$, $\sigma^2 = s_z^2$. In contrast, maximizing $L_1$ with respect to $\theta$ and $z_m$ yields a common estimate $\bar{z}$ for all components of $z_m$, and estimates $\mu = \bar{z}$, $\sigma^2 = s_z^2(n/N)$. Thus the maximum likelihood estimate of the mean is obtained, but the maximum likelihood estimate of the variance is multiplied by the fraction of observed data. When the fraction of missing data is substantial (for example, $n/N = 0.5$), the estimated variance $\sigma^2$ is badly biased, and this bias does not vanish as $N \to \infty$ unless $n/N \to 0$; more relevant asymptotics would fix $n/N$ as the sample size increases.

## Example 2. Missing Plot Analysis of Variance

Suppose we add to the previous example a set of covariates $x$ which is observed for all $N$ observations. We assume that the value of $z$ for observation $i$ with covariate values $x_i$ is Normal with mean $\beta_0 + \beta^T x_i$ and variance $\sigma^2$. The estimates of $\beta_0$ and $\beta$ obtained by maximizing $L_1$ are the maximum likelihood estimates, obtained by least squares regression with the $n$ observed data points. However, as in Example 1, the estimate of variance is the maximum likelihood estimate multiplied by the proportion of observed values.

These results provide one justification for the analysis of missing plots in analysis of variance designs mentioned in section 1: jointly estimating the values of the outcome variable for the missing plots and the parameters leads to maximum likelihood estimates of the effects $\beta$. However an adjustment is needed to the resulting estimate of the residual variance $\sigma^2$, as the literature on missing plot analysis explicitly recognizes.

## Example 3. An Exponential Sample

In the first two examples estimation based on maximizing $L_1$ at least yields reasonable estimates of location, even though estimates of the scale parameter need adjustment. However in other examples, estimates of location can also be biased. For example, consider a censored sample from an exponential distribution with mean $\mu$, where $z_p$ represents the $n$ observed values which lie below a known censoring point $c$, and $z_m$ represents the $N-n$ values beyond $c$ which are censored. The maximum likelihood estimate of $\mu$ is $\hat{\mu} = \bar{z} + (N-n)c/n$. Maximization of $L_1$ leads to estimating censored values of $z$ at the censoring point $c$, and estimating $\mu$ by $(n/N)\hat{\mu}$. Thus in this case the estimate of the mean is inconsistent unless the proportion of missing values tends to zero as the sample size increases.

Example 4. A Bivariate Normal Sample with Missing Predictor Variables.

Biased estimates of location parameters can also occur in problems involving the normal distribution. For example, suppose that $z_i = (x_i, y_i)$ $i = 1,\ldots,N$ are $N$ observations from a bivariate normal distribution with mean $(\mu_x, \mu_y)$, variances $\sigma_x^2$ and $\sigma_y^2$, and correlation $\rho$, where $y_i$ is observed for all $N$ observations, and $x_1,\ldots,x_n$ are observed but $x_{n+1},\ldots,x_N$ are missing at random. Suppose that interest is focussed on the regression coefficient of $y_i$ on $x_i$, $\beta_{y.x} = \rho\sigma_y/\sigma_x = \beta_{x.y}\,\sigma_y^2/\sigma_x^2$. The maximum likelihood estimate of $\beta_{y.x}$ is

$$\hat{\beta}_{y.x} = \hat{\beta}_{x.y}\hat{\sigma}_y^2/\hat{\sigma}_x^2 \quad ,$$

where $\hat{\beta}_{x.y} = \sum_{i=1}^{n} (x_i-\bar{x})y_i / \sum_{i=1}^{n} (x_i-\bar{x})^2$, $\bar{x} = \sum_{i=1}^{n} x_i$; $\hat{\sigma}_y^2 = N^{-1}\sum_{i=1}^{N} (y_i-\bar{y})^2$, $\bar{y} = N^{-1}\sum_{i=1}^{N} y_i$; and $\hat{\sigma}_x^2 = \hat{\beta}_{x.y}^2\hat{\sigma}_y^2 + n^{-1}\sum_{i=1}^{n} (x_i-\hat{\beta}_{x.y}y_i)^2$.

*Maximization of* (2) over parameters and data yields for estimated $\beta_{y.x}$,

$$\hat{\hat{\beta}}_{y.x} = \hat{\beta}_{y.x}\hat{\sigma}_x^2/\hat{\hat{\sigma}}_x^2 \quad ,$$

where $\hat{\hat{\sigma}}_x^2 = \hat{\beta}_{x.y}^2\hat{\sigma}_y^2 + N^{-1}\sum_{i=1}^{n} (x_i-\hat{\beta}_{x.y}y_i)^2$. The estimate $\hat{\hat{\beta}}_{y.x}$ can be badly biased, and again this bias persists as $N \to \infty$ unless the fraction of missing observations tends to zero.

This example is a special case of the problem considered by Press and Scott (1976). They observe that for the general problem they considered their estimates based on maximizing $L_2$ are consistent only if the fraction of missing observations tends to zero. The correct maximum likelihood approach, as discussed by Trawinski and Bargman (1964), Hartley and Hocking (1971), Orchard and Woodbury (1972), Beale and Little (1975) and Dempster, Laird and Rubin (1977) leads to estimates which are consistent as the sample size increases with the fraction of missing data held constant.

## 3. Missing Values as Parameters

Both maximum likelihood and the maximization of $L_1$ over parameters and missing data assumes the existence of a model that specifies a distribution for the observed and missing values of $z$. Occasionally it is possible that situations will arise when it may be desirable to avoid specifying a distribution for the missing values and to treat them as genuine unknown parameters. Hartley and Hocking (1971, section 4 and 5) discuss the regression of $y_i$ on $x_i$, where the values $x_i$ correspond to fixed points in an experimental design, $y_i$ is observed for all units $i$ and components of $x_i$ are missing for some units. Writing $x_p$ and $x_m$ for the present and missing values of $x$, respectively, Hartley and Hocking (1971) suggest drawing inferences by maximizing the complete data likelihood based on the conditional distribution of $y$ given $x$

$$L_3(\theta, \tilde{x}_m | \tilde{y}, \tilde{x}_p) = f(y | \tilde{x}_m, \tilde{x}_p; \theta) \tag{4}$$

with respect to $x_m$ and the parameters $\theta$. Hartley and Hocking discuss analyses where values of $x_m$ are unconstrained or are constrained to be any of $k$ alternatives. We believe that in most practical situations it is more natural to include a distribution for the missing values in the model (Rubin, 1971). From a strict likelihood perspective, however, there is no reason in principle to reject inferences based on (4). The question of whether $x_m$ should be treated as fixed or integrated out of the likelihood (as in (2)) relates to the more general issue of statistical inference in the presence of nuisance parameters, which lies outside the scope of this note.

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